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THE ZHUKOVSKII FUNCTION AND SOME PROBLEMS IN FILTRATION THEORY†

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The contribution of N. Ye. Zhukovskii to filtration theory is noted. Two problems concerning a prolate cycloid and a curtate cycloid, the solutions of which can be obtained using a function introduced by Zhukovskii, are considered as an example.

The great Russian scientist Nikolai Yegorovich Zhukovskii is mainly known for his work in aerodynamics and hydrodynamics. However, he also investigated [1–3] the "theory of ground waters", that is, filtration theory and, together with the papers [1, 2], there is also a paper of his [3] which was only published posthumously in 1923.

Zhukovskii extensively used the methods of the theory of a complex variable in hydrodynamics and aerodynamics and he introduced a function θ which became known as the Zhukovskii function and, up to the present day, enables a number of problems in filtration theory to be solved. This function of a complex variable $z = x + iy$, where x and y are the Cartesian coordinates of a point, has the form

$$\theta = \omega - i\varkappa z, \qquad \varkappa = \text{const} \tag{1}$$

Here $\varkappa$ is the filtration coefficient of the ground, which is assumed to be homogeneous, $\omega = \varphi + i\psi$ is the complex potential φ is the velocity potential and ψ is the stream function.

On separating the real and imaginary parts in (1), we obtain

$$\theta = \theta_1 + i\theta_2, \quad \theta_1 = \varphi + \varkappa y, \quad \theta_2 = \psi - \varkappa x \tag{2}$$

Note that the Zhukovskii function θ must have the form of (1), assuming that the y axis is directed vertically upwards, and the velocity potential has the form [4]

$$\varphi = -\varkappa h = -\varkappa(p/(\rho g) + y) \tag{3}$$

where p is the pressure, ρ is the fluid density, g is the acceleration due to gravity and h is the pressure head. If, however, the y axis is directed vertically downwards, the signs of the quantities z and y in formulae (1) and (3) must be changed.

To fix our ideas, we shall assume that the y axis is directed vertically upwards.

It is seen from formulae (2) and (3) that the expression

$$\theta_1 = \varphi + \varkappa y = -\varkappa p/(\rho g) \tag{4}$$

only differs from the pressure p by a constant factor.

The expression $P = p + ip'$, where

$$\theta_2 = -\varkappa p'/(\rho g) \tag{5}$$

may be called the complex pressure.

The formula

$$\theta = -\varkappa P/(\rho g)$$

then follows from (1), (4) and (5).

The Zhukovskii function θ is therefore proportional to the complex pressure P in homogeneous ground.

At the free surface of ground flow, the pressure is equal to the atmospheric pressure. When there is capillarity, the pressure is also constant and, if it is assumed that the atmospheric pressure is equal to zero, on the boundary of the capillary edge $p = -\rho g h_c$, where h_c is the height of the capillary rise of the water in the ground.

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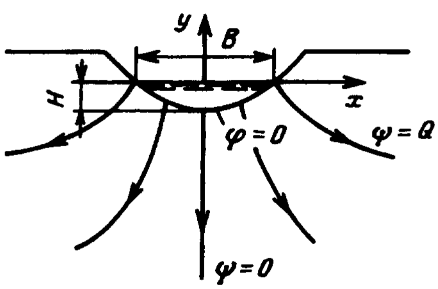

Fig. 1.

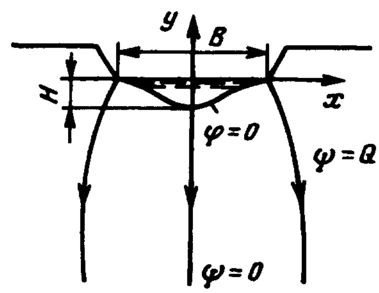

Fig. 2.

By using the function θ, defined by formula (1), Zhukovskii himself solved problems on a Zhukovskii channel and a self-contained drain (a Zhukovskii drain) [3].

Here, as an example of the use of the Zhukovskii function, we shall dwell on Kozeny channels with a curvilinear profile and consider two problems concerning a prolate cycloid and a curtate cycloid.

These two channels can be obtained by assuming that the Zhukovskii function has the form [4, 5]

$$\theta = -\varkappa H \exp\left(\pm \frac{\pi\omega}{2Q}\right), \quad \varkappa = \text{const} \tag{6}$$

where the plus sign is taken in the case of a prolate cycloid and the minus sign in the case of a curtate cycloid. The function θ is defined by formula (1) and H is the maximum depth of the channel.

Separation of the real and imaginary parts in formulae (1) and (6) gives

$$x = \frac{\psi}{\varkappa} \pm H \exp\left(\pm \frac{\pi\varphi}{2Q}\right) \sin \frac{\pi\psi}{2Q}$$

$$y = -\frac{\psi}{\varkappa} - H \exp\left(\pm \frac{\pi\varphi}{2Q}\right) \cos \frac{\pi\psi}{2Q}$$

We obtain the equation of the right-hand branch of the free surface by putting $\psi = Q$, $\varphi = -\varkappa y$ (Figs 1 and 2) and

$$x = \frac{Q}{\varkappa} \pm H \exp\left(\mp \frac{\pi \varkappa y}{2Q}\right)$$

Substituting $\varphi = 0$ into (6), we find the curves which can be taken as the contour of the cross-section of the channel

$$x = \frac{\psi}{\varkappa} \pm H \sin \frac{\pi\psi}{2Q}, \quad y = -H \cos \frac{\pi\psi}{2Q} \tag{7}$$

On eliminating the parameter ψ from Eqs (7), we obtain the relations

$$x = \pm\sqrt{H^2 - y^2} + \frac{2Q}{\varkappa\pi} \arccos \frac{y}{H} \tag{8}$$

The upper sign gives a prolate cycloid (Fig. 1) and the lower sign a curtate cycloid (Fig. 2). Here, the flow is considered in the right-hand half-plane and Q is the discharge from half the channel. In order to obtain the relation between the discharge and the other characteristics of the problem, we put $x = B/2$, $\varphi = 0$, $\psi = Q$ (B is the channel width (Figs 1 and 2)) in (6). Then, for the overall discharge from the channel, we obtain

$$2Q = \varkappa(B \mp 2H) \tag{9}$$

and, from (1) and (6), we find

$$\frac{dz}{d\omega} = -\frac{H\pi i}{2Q} \exp\left(\pm \frac{\pi\omega}{2Q}\right) - \frac{i}{\varkappa} \tag{10}$$

For a prolate cycloid (the case with the upper sign), it is clear from (9) and (10) that the solution only makes sense when $B > 2H$. This is a flow with a backwater, the velocity of which is equal to zero at infinity, which follows from (10) when $\varphi \to \infty$. The case of a curtate cycloid (the lower sign and motion without a backwater) produces greater diversity [6, 7].

Note that the problem involving a curtate cycloid can be considered as an example, from filtration theory [8], of the problem of the conformal mapping of a circular tetragon into the upper half-plane in the limiting case [9], namely, of a circular triangle, two sides of which are equal and in which all three angles are right angles.

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